Continuous and discrete stable processes

W. H. Lee, K. I. Hopcraft, and E. Jakeman
School of Mathematical Sciences, Applied Mathematics Division, University of Nottingham, Nottingham, NG7 2RD, United Kingdom
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The one-sided Lévy-stable probability densities and the discrete-stable distributions form a doubly stochastic Poisson transform pair. This relationship facilitates the formulation of a class of continuous-stable stochastic processes.

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The concept of statistical stability underpins classical statistical mechanics in general and the behaviors presented by complex systems in particular. Stability refers to the mathematical property whereby sums of identical and stably distributed random variables are similarly distributed [1], the Gaussian random variables being the most familiar example of these. Other members of the class are characterized by probability density functions $p(x)$ that have power-law tails, such that $p(x)\sim|x|^{-(1+\nu)}$ with index in the range $0<\nu<2$ and hence the *variance* and higher moments of these distributions are infinite. Stability has a wider currency, however, because of its impact on prescribing the behavior of sums of random variables of arbitrary distribution. The central limit theorem of classical statistics and its generalizations [2] stipulate the limiting distribution for sums of a large number of random variables, this being the Gaussian when the variances of the summands all exist and one of the other members of the stable class when at least one of the variances is infinite. The concept of statistical stability can be extended to encompass discrete random variables too [3,4]. Here the analog of the Gaussian random variable is assumed by Poisson-distributed random variables, while the power-law distributions for the random integer $n\geq 0$ adopt the asymptotic form $P(n)\sim 1/n^{1+\nu}$ where the index is now in the range $0<\nu<1$, so that the *mean* and higher moments of these distributions do not exist. The equivalent generalization of the discrete version of the central limit theorem [5] can be paraphrased as follows: the limiting distribution for sums of a large number of discrete random variables is Poisson when the means of all the summands exist and one of the other members of the class when at least one of the means is infinite. The evident similarities between the discrete and continuous-stable random variables and the attendant generalized limit theorems prompts investigating whether there exists a more fundamental connection between them. The first purpose of this paper is to establish precisely the nature of this interrelationship, which is that the discrete and one-sided continuous-stable distributions form a Poisson transform pair. Although this relationship has been noted previously in an abstract context [6], the conditional relationship between the scaling constants appearing in the transform pairs was not recognized. The proof given here identifies explicitly the dependence of the scaling constants with the index ν, and this association is critical for this paper's second aim, to formulate the important generalization from stable-distributed variables to well-defined continuous processes.

The continuous Lévy-stable distributions are defined through their characteristic function $C(u)$, which is the Fourier transform of the probability density function (PDF)

$$C(u)=\int_{-\infty}^{\infty}p(x)\exp(iux)dx$$
$$=\exp[-a|u|^{\nu}[1-i\beta\,\mathrm{sgn}(u)\Phi(u,\nu)]],$$
$$a>0,\quad |\beta|\leq 1,\quad 0<\nu\leq 2, \tag{1}$$

where

$$\Phi(u,\nu)=\begin{cases}\tan\left(\dfrac{\pi\nu}{2}\right), & \nu\neq 1,\\ -\dfrac{2}{\pi}\ln|u|, & \nu=1\end{cases}$$

[1,7]. In the above a is a scaling constant. The symmetry of the distribution is controlled by the parameter β: when $\beta=0$, the distribution is symmetric and defined for all x. By contrast when $|\beta|=1$ and the index falls in the range $0<\nu<1$, the distributions are "one sided" on the half-line defined through $\mathrm{sgn}(x)=\beta$. The index ν describes the power-law behavior for large x. When $\nu=2$ the distribution is a Gaussian with variance $2a$. The PDF is found by Fourier inversion of the characteristic function, and apart from a few special cases (e.g., [7]), closed-form expressions for the distributions cannot be found.

The discrete stable distributions are defined through the generating function

$$Q(s)=\sum_{n=0}^{\infty}(1-s)^{n}P(n)=\exp(-As^{\nu}), \tag{2}$$

where $0<\nu\leq 1$. Here A is the analogous scaling constant that can be identified with the mean of n when $\nu=1$, in which case the generating function is that for the Poisson distribution. Only exceptionally can a closed form expression be found for the distribution [4] but

$$P(n)=\frac{(-1)^{n}}{n!}\left.\frac{\partial^{n}Q(s)}{\partial s^{n}}\right|_{s=1}$$

can always be used. The Poisson transform describes how a discrete random variable n with probability distribution $P(n)$ can be generated by the action of some underlying continuous fluctuation with density $p(x)$,

$$P(n) = \frac{1}{n!}\int_0^\infty x^n \exp(-x)p(x)dx, \quad (3)$$

and was introduced as the representation of a doubly stochastic process with reference to the breakdown of looms [8], having since found application to the photoelectric detection of photons [9] among many others (e.g., [10]). The Poisson transform requires that the continuous random variable is defined for $x \geqslant 0$ and the continuous-stable distributions with this property are those for which $\beta = 1$, in which case $0 < \nu < 1$, and this range of ν coincides with that for which the discrete stable distributions are defined.

Supposing that a discrete distribution has a Poisson transform representation given by Eq. (3), it follows that the generating function as defined in Eq. (2) is the Laplace transform of the continuous density $p(x)$ because

$$Q(s) = \sum_{n=0}^{\infty} \frac{(1-s)^n}{n!}\int_0^\infty x^n \exp(-x)p(x)dx$$
$$= \int_0^\infty dx p(x)\exp(-sx) = L[p] = \frac{1}{2\pi}\int_{-\infty}^{\infty} du \frac{C(u)}{s+iu}, \quad (4)$$

the last expression resulting from writing the density in terms of its characteristic function and performing the integral over the x variable. The result (4) is general, but on specializing to the characteristic function appropriate for the one-sided stable distributions one obtains

$$Q(s) = \frac{1}{2\pi}\int_0^\infty du \frac{\exp(-au^\nu)}{(s+iu)(s-iu)} \times \left\{ (s-iu)\exp\left[iau^\nu \tan\left(\frac{\pi\nu}{2}\right)\right] + \text{c.c.}\right\},$$

which may be evaluated using a complex contour Γ comprising the positive x axis extending from the origin to the point X, followed by the arc γ_1 of a circle connecting X to $X\exp(i\Theta)$ and closed by γ_2, a straight line back to the origin. Employing the residue theorem and noting that the only singularity of the integrand occurs where $u = is$ provided that contour is closed along a ray where $\pi/2 \leqslant \Theta < \pi$, it follows that

$$Q(s) = \frac{1}{2\pi}\left[2\pi i \text{Res}\left(\frac{\exp(-au^\nu)}{(s+iu)(s-iu)} \times \left\{(s-iu)\exp\left[iau^\nu \tan\left(\frac{\pi\nu}{2}\right)\right] + \text{c.c.}\right\}_{u=is}\right)\right] - I_1 - I_2,$$

where I_1 and I_2 are the contributions from the paths γ_1 and γ_2. It is straightforward to show that these contributions vanish as $X \to \infty$ in the limit as $\Theta \to \pi$, and the residue can be evaluated to obtain

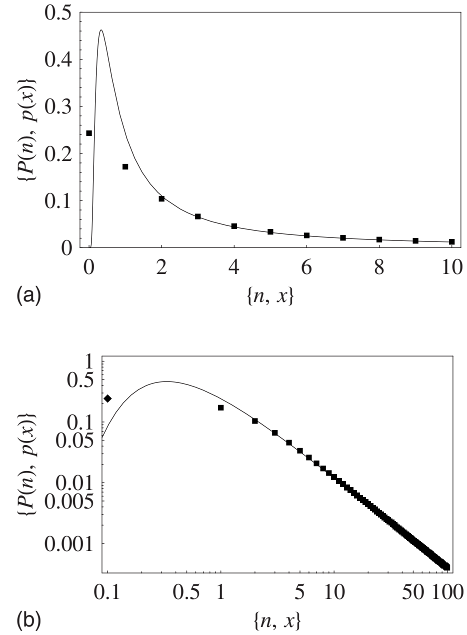

FIG. 1. Comparison between the continuous- (solid line) and discrete- (squares) stable distributions with index $\nu = 1/2$ and $a = 1$ shown in linear (a) and log-log plots (b).

$$Q(s) = \exp\left\{-a(is)^\nu\left[1 - i\tan\left(\frac{\pi\nu}{2}\right)\right]\right\}$$
$$= \exp\left(-a\exp\left(\frac{i\pi\nu}{2}\right)s^\nu\left\{\left[1+\tan^2\left(\frac{\pi\nu}{2}\right)\right]^{1/2} \times \exp\left(-\frac{i\pi\nu}{2}\right)\right\}\right) = \exp\left[-a\sec\left(\frac{\pi\nu}{2}\right)s^\nu\right],$$

which upon comparison with Eq. (2) shows that the one-sided continuous Lévy-stable distributions and the discrete stable distributions form a Poisson transform pair with

$$A = a\sec(\pi\nu/2). \quad (5)$$

It should be stressed that this result is valid only for $0 < \nu < 1$ and cannot be extended to the case $\nu = 1$ because the continuous distribution corresponding to this is defined over the entire real line rather than positive values alone. Moreover, recall that the discrete distribution with $\nu = 1$ is the Poisson, corresponding to $p(x)$ given by a delta function $\delta(x - \bar{n})$.

Figure 1(a) compares the discrete and continuous distributions for when $\nu = 1/2$, in which case

$$P(n) = \frac{2}{\pi^{1/2}n!}A^{n+1/2}K_{n-1/2}(2A)$$

and

$$p_{1/2}(x) = \frac{a}{2\pi^{1/2}x^{3/2}}\exp\left(-\frac{a^2}{4x}\right), \quad (6)$$

where $K_n(x)$ is a modified Bessel function of the second kind [11]. The discrete distribution is monotonically decreasing

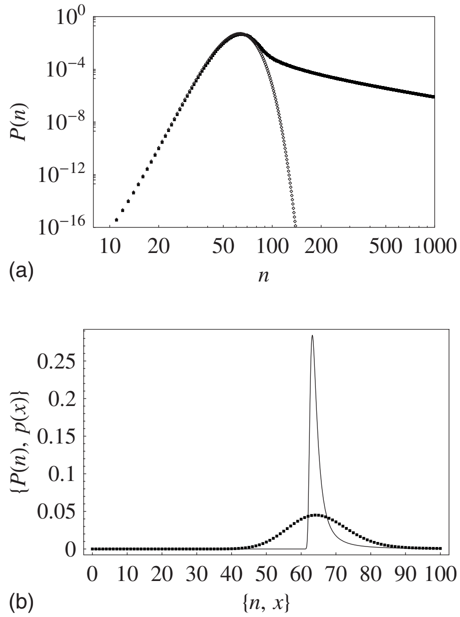

FIG. 2. (a) Comparison between the Poisson and discrete-stable distribution with $\nu=0.99$, the power-law tail is evident in the latter. (b) Comparison between the discrete- and continuous-stable distributions with $\nu=0.99$. Both have power-law tails with the same index which is not discernible on the graphs.

with $P(0)=\exp(-A)$, whose value is shown by the diamond-shaped character in the figure. By contrast the continuous distribution has an acclivity in the vicinity of the origin, where $p_{1/2}(0)=0$. The logarithmic plot Fig. 1(b) illustrates the power-law tail of both distributions.

For small values of n and with $\nu\approx 1$, the Poisson distribution and the discrete-stable distributions are almost identical, however: for $n \gg 1$ a power-law tail in the latter becomes the distinguishing feature. This is illustrated in Fig. 2(a) where the two distributions are contrasted for when for $a=1$ and $\nu=0.99$—the distributions only differ discernibly for $n>100$. Thus, as $\nu\to 1$, for values of $n<a\sec(\pi\nu/2)$ the discrete-stable distributions behave like a Poisson distribution with mean $a\sec(\pi\nu/2)$, whereas for values of n greater than this, the power-law tail becomes established. Comparing the discrete-stable distribution and its continuous-stable counterpart (with $a=1, \nu=0.99, \beta=1$), the differences between the distributions are more pronounced. The continuous distribution becomes more peaked in nature, but it should be borne in mind that it still has a power-law tail and so is not localized like the δ function it begins to resemble and finally attains when $\nu=1$. It is worthwhile noting that the discrete-stable distributions can be obtained from a family of Markovian stochastic processes [4]. When ν is close to unity, these processes have the characteristics of what is ostensibly a Poisson process but with extreme values deriving from the power-law tails; these rare events are necessarily part of the process. Thus the discrete-stable processes can be utilized as a model for the prediction of rare "outlying" events.

So far, only doubly stochastic stable *variables* have been discussed: however, it is also possible to generalize the doubly stochastic approach to the relationship between discrete and continuous stable *processes*. Thus the partial differential equation for a process that has Eq. (2) as its stationary solution can be Laplace inverted to obtain the Fokker-Planck equation for $p(x,t)$. One such process is a population model consisting of deaths and multiple immigrations:

$$\frac{dP(n,t)}{dt} = \mu[(n+1)P(n+1,t) - nP(n,t)] - P(n,t)\sum_{m=1}^{\infty}\alpha_m + \sum_{m=1}^{n}\alpha_m P(n-m,t). \qquad (7)$$

The first two terms on the right-hand side of this equation correspond to depletions due to single deaths occurring at a rate μ, and the second two represent multiple immigrations into the population occurring with rates $\alpha_m = A\mu\nu^2\Gamma(m-\nu)/m!\Gamma(1-\nu)$. The corresponding generating function equation is [4,12]

$$\frac{\partial}{\partial t}Q(s,t) = -\mu s\frac{\partial}{\partial s}Q(s,t) - \nu\mu A s^{\nu}Q(s,t),$$

which has Eq. (2) as its stationary solution and is similar in form to that obtained for the characteristic function of continuous Lévy noise [13], which is defined for fluctuations occurring for *all* values of x. Inverse Laplace transformation gives the integro-differential equation

$$\frac{\partial}{\partial t}p(x,t) = \mu\frac{\partial}{\partial x}[xp(x,t)] + \frac{A\mu\nu}{\Gamma(1-\nu)}\int_0^x \frac{dx'}{(x-x')^{\nu}}\frac{\partial p(x',t)}{\partial x'}, \qquad (8)$$

the convolution revealing explicitly the nonlocal and causality effects influencing the continuous process. Equation (8) is distinct from those appearing in [14] that were developed to treat continuous-stable processes for all values of x.

The solution of Eq. (8) is the conditional continuous density related to the conditional discrete distribution of Eq. (7) via the doubly stochastic representation (3) and the joint density can be calculated from this solution in the usual way. However, note that in order to calculate the joint generating function for the continuous process *directly* from an expression for the joint generating function for the discrete process requires the elimination from the latter of terms arising solely from the discrete nature of the variable. For example, the joint generating function for the discrete stable process formed by the above death-multiple immigration population model is [12]

$$Q(s,s') = \exp(-A\{s'^{\nu}[1-\theta(t)^{\nu}] + [s + (1-s)s'\theta(t)]^{\nu}\}), \qquad (9)$$

where $\theta(t)=\exp(-\mu t)$. Comparison with the joint-Poisson process, whose continuous analog must be a product of two uncorrelated δ functions, demands that the scaling in s required to achieve the correct dense limit must necessarily suppress the term involving $ss'\theta(t)$. Indeed, comparisons with discrete processes whose moments and normalized correlation function $\langle n(0)n(t)\rangle/\bar{n}^2$ exist (e.g., [15]) reveals that terms involving $ss'\theta(t)$ appearing in the generating function are responsible for contributions of order $\bar{n}^{-1}$ in the normalized correlation function. These dependences are an intrinsic feature of discrete processes that are not present in their continuous analogs. From Eq. (5), as $\nu \to 1$, $A \to \bar{n} \to \infty$, and so the $O(\bar{n}^{-1})$ terms vanish. The scaling of Eq. (9) that achieves this suppression for *all* allowable values of ν is $s \to (a/A)^{1/\nu} s$ which upon letting $A \to \infty$ gives

$$Q(s,s') = \exp(-a\{s'^{\nu}[1-\theta(t)^{\nu}] + [s+s'\theta(t)]^{\nu}\}). \quad (10)$$

Equation (10) is therefore the double-Laplace transform of the one-sided continuous-stable *process*. The same result obtains when considering the birth–death–multiple-immigration process [12] which has a different generating function to that given by Eq. (9). Indeed result (10) is a valid joint-generating function corresponding to a stable single-interval variable for any function of θ [provided that $0 < \theta(t) \leq 1$ and decreases away from the origin] and can be Laplace inverted to obtain the general joint-stable density:

$$p(x_1,x_2) = \frac{1}{a^{2/\nu}(1-\theta^{\nu})^{1/\nu}} p_\nu\left(\frac{x_1}{a^{1/\nu}}\right) p_\nu\left(\frac{x_2 - x_1\theta}{a^{1/\nu}(1-\theta^{\nu})^{1/\nu}}\right) H(x_2 - x_1\theta),$$

where $L^{-1}[\exp(-s^{\nu})] = p_\nu(x)$ is the marginal one-sided Lévy density function and $H(z)$ is the Heaviside step function.

This paper has shown that the class of discrete stable variables may be generated from a subset of the continuous Lévy-stable variables through a doubly stochastic Poisson transform. This result has enabled the asymptotic behavior near the Poisson limit (or, in the case of a continuous process, near the δ-function distribution limit) to be elucidated, establishing the role and importance of outlying events and extreme behavior. The doubly stochastic representation has been extended to stable *processes*, and results for joint generating functions and joint-stable densities have been derived. This facilitates the analysis and simulation of both discrete- and continuous-stable processes that are important in the modeling of many natural phenomena that have extremal and/or fractal characteristics.

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